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# Dispersion Analysis of Nonlocal Elastic Waves in Thin Plates: Nonlocal Elastic Waves Dispersion in Thin Plates

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## ABSTRACT

This research investigates the propagation of elastic waves in uniform thin plates using the framework of nonlocal elasticity theory. Focusing on plates with stress-free boundary surfaces, the study derives and analyses two distinct dispersion relations corresponding to symmetrical and antisymmetrical wave modes with respect to the plate's mid-plane. Owing to this symmetry, the plate can effectively be treated as two identical halves, which simplifies the mathematical analysis while preserving the essential physical behaviour of the system. The study further examines the influence of the nonlocality parameter on the dispersion characteristics of both wave modes, showing that small-scale effects significantly modify wave propagation compared with predictions from classical elasticity theory. In addition to the general formulation, the paper also considers several limiting cases to validate the model and clarify its physical implications under special conditions. Numerical computations and graphical illustrations are included to provide a clearer picture of how nonlocal effects alter wave behaviour, emphasizing their specific role in shaping the dispersion response of thin plate structures.

## 1. Introduction

Under the framework of classical continuum mechanics, the stress state at a particular material point is derived exclusively from the local strain at that exact point. This local theory assumes a purely localized material response, neglecting any microstructural interactions across space. In contrast, Eringen (1983, 2002) introduced the theory of nonlocal elasticity, which posits that the mechanical state at a given reference point is influenced by the collective behavior of all other particles throughout the elastic solid body. Because real materials are not perfectly continuous at microscopic scales, nonlocal theory assumes that stress at a reference point depends on the strain field over a nearby surrounding region. By incorporating long-range interatomic and molecular interactions into the governing equations

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via a characteristic length-scale parameter, this advanced framework successfully expands upon conventional elasticity. The foundational principles and general formulations of nonlocal continuum field theories were established in early seminal works by Edelen et al. (1971) and Eringen and Edelen (1972). Over the years, these concepts have been extensively applied and extended to analyze complex wave propagation profiles, microstructural oscillations, and scaling dependencies (Gopalakrishnan & Narender, 2013; Nowinski, 1984, 1993; Reddy & Ramabrahmam, 2010).

Historically, the study of guided wave phenomena in bounded configurations began with classical elasticity. Lord Rayleigh (1885) first mathematically detailed the propagation of surface waves along the boundary of a semi-infinite solid, while Lamb (1917) pioneered the physical theory predicting guided wave modes in free elastic plates. The theoretical depth and experimental validity of these classic elastodynamic principles have since been firmly established by Ewing et al. (1957), Viktorov (1967), and Graff (1975). In modern structural design and health monitoring regimes (Pant, 2014), evaluating these guided waves at micro- and nano-scales requires scale-dependent modeling. To address this, Liu and Yang (2012) formulated a nonlocal model to derive the dispersion relations of Lamb waves in double-layered nanoplates. Chebako et al. (2017) employed a non-local asymptotic theory to model thin elastic plates, providing low-frequency, long-wavelength approximations for bending and longitudinal motions. Furthermore, nonlocal wave propagation phenomena have been extensively studied across diverse microstructural mediums, including micropolar solid half-spaces (Khurana & Tomar, 2017) and elastic solids featuring distributed micro-voids (Kaur et al., 2018; Singh et al., 2017).

As nonlocal modeling matured, a critical body of literature focused on refining these models to address mathematical inconsistencies at material boundaries, often termed "boundary condition paradoxes" (Barretta et al., 2023; Ceballes et al., 2021). To resolve these anomalies, researchers have investigated the deep analytical well-posedness of Eringen's integral and differential frameworks for harmonic plane waves (Pham & Vu, 2024), while implementing dual-parameter and modified gradient approaches to maintain physical consistency (Vinh & Tuan, 2023). This mathematical refinement has enabled the precise modeling of scale-dependent dynamics in complex, multi-field, and smart materials.

For instance, recent studies examine shear-horizontal and plane waves passing through layered nanostructures under the influence of environmental coupling, micro-voids, and thermal stresses (Kaur et al., 2019; Kumar & Tomar, 2020; Tian et al., 2020). Modern engineering applications have pushed these theories further into the domain of advanced structural design, optimizing wave control via nonlocal elastic metasurfaces (Zhu et al., 2020), mapping propagating and evanescent modes in functionally graded nanoplates (Liu et al., 2023), and assessing multi-field thermomechanical responses in porous, graphene-reinforced magneto-electro-elastic nanoshells and sandwich nanoplates (Aktaş, 2024; Wang et al., 2023). Advanced matrix methods, such as the transfer matrix approach, have also been introduced to systematically capture Lamb wave characteristics across weakly nonlocal layered composite plates (Anh et al., 2025).

This paper investigates guided wave propagation in an infinite, isotropic thin plate within the strict mathematical context of linear nonlocal elasticity. The non-classical equations of motion are systematically established, and explicit dispersion relations are derived for a macroscopically uniform plate bounded by stress-free surfaces. The analysis confirms that, mirroring the classical paradigm, the wave field cleanly decomposes into two decoupled families of modes: symmetric and antisymmetric. However, the results demonstrate that both mode families are significantly altered by the introduction of the nonlocality parameter. In the short-wave limit, the symmetric and antisymmetric modes converge asymptotically to a unified Rayleigh-Lamb-type frequency equation; yet, unlike the non-dispersive classical limit, this surface-wave equation becomes inherently dispersive due to the nonlocal length scale. In the long-wave limit, the symmetric modes separate into distinct transverse and longitudinal compressional (plate) waves. Notably, the resulting plate-wave equation is explicitly modified by the nonlocality parameter, introducing a physical dispersion completely absent in the classical model. Numerical simulations conducted for a benchmark aluminum plate indicate that even minor nonlocality factors noticeably influence Lamb-wave velocity profiles and phase frequencies. These physical scale effects are illustrated and discussed through comprehensive dispersion curves for both symmetric and antisymmetric modes, explicitly highlighting the role of material nonlocality in high-frequency structural acoustics.

## 2. Non-Local Stress and Equation of Motion

In the context of Eringen's nonlocal elasticity theory (Eringen, 1983, 2002), the nonlocal stress tensor $\tau_{ij}$ at a point $y$ is expressed as a weighted average of the classical local stress tensor $\sigma_{ij}$ over the entire body domain $\Omega$. The constitutive relation is written as

$$\tau_{ij}(y) = \int_{\Omega} k(|y' - y|, \eta)\sigma_{ij}(y')d\Omega(y')k, \quad i, j = 1,2,3. \tag{1}$$

where $\Omega$ denotes the region occupied by the body, $k(|\mathbf{y}'-\mathbf{y}|,\eta)$ is the nonlocal kernel function, and $|\mathbf{y}'-\mathbf{y}|$ represents the Euclidean distance between the source point $\mathbf{y}'$ and the field point $\mathbf{y}$. The parameter $\eta$ is the non-dimensional nonlocal length-scale parameter, defined as $\eta = \frac{m_0 \ell_1}{\ell_2}$, where $m_0 \ell_1$ is the material-dependent internal characteristic length and $\ell_2$ is the relevant external characteristic length of the body. Equivalently, if the governing equations are expressed in dimensional coordinates, the dimensional nonlocal length may be denoted by $\ell = m_0 \ell_1$. The differential operator associated with the kernel is written as $\Gamma = 1 - \ell^2 \nabla^2$, where $\nabla^2$ is the Laplacian operator. This form is dimensionally consistent because $\ell^2 \nabla^2$ is dimensionless. The kernel function when operated by the linear operator $\Gamma$, such that $\Gamma k(|\mathbf{y}'-\mathbf{y}|,\eta) = \delta(\mathbf{y}'-\mathbf{y})$.

Using this property of the kernel function, the integral constitutive relation in equation (1) can be reduced to the following differential form:

$$\Gamma \tau_{kl} = \sigma_{kl} \tag{2a}$$

In the absence of body forces, the macroscopic equation of motion is governed by the divergence of the actual nonlocal stress tensor:

$$\tau_{kl,k} = \rho \ddot{u}_l \tag{2b}$$

where $u_l$ represents the displacement vector, $\rho$ is the material density, over-dots denote material time derivatives, and the subscript comma signifies partial differentiation with respect to spatial coordinates ($x_k$). Applying the operator $\Gamma$ to equation (2b) yields the equations of motion written in terms of classical local stresses:

$$\sigma_{kl,k} = \Gamma\left(\rho \ddot{u}_l\right) \tag{3}$$

For a linear, isotropic, elastic solid medium, the classical local stress tensor $\sigma_{kl}$ satisfies Hooke's law relative to the stiffness coefficients $C_{pq}$ and infinitesimal strain components $e_{kl}$:

$$\begin{aligned}
&\sigma_{11} = C_{11}e_{11} + C_{12}e_{22} + C_{12}e_{33} \\
&\sigma_{22} = C_{12}e_{11} + C_{11}e_{22} + C_{12}e_{33} \ , \\
&\sigma_{33} = C_{12}e_{11} + C_{12}e_{22} + C_{11}e_{33}, \\
&\sigma_{23} = 2C_{44}e_{23}, \ \sigma_{13} = 2C_{44}e_{13}, \ \sigma_{12} = 2C_{44}e_{12},
\end{aligned} \tag{4}$$

where the symmetric linear strain tensor is defined as:

$$e_{kl} = \frac{1}{2}\left(u_{k,l} + u_{l,k}\right) \tag{5}$$

Relation among elastic constants $C_{11}, C_{12}, C_{44}$ and $E$ (Young's modulus) and $\eta$ (Poisson's ratio) are:

$$C_{11} = \frac{E(1-\mu)}{1-\mu-2\mu^2}, \ C_{12} = \frac{\mu E}{1-\mu-2\mu^2} \text{ and } C_{44} = \frac{C_{11}-C_{12}}{2}.$$

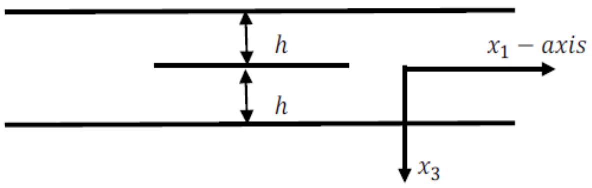

*Figure 1. Geometry of the problem*

## 3. Model of the Problem and Solution

We consider a plate of infinite extent and finite thickness $2h$ composed of homogeneous isotropic elastic solid. Take the origin of the coordinate system $\left(x_1, x_2, x_3\right)$ on the middle surface of the plate. The $x_1 - x_2$ plane is chosen to coincide with the middle plane and $x_3$-axis normal to it along the thickness of the plate. The top and the bottom surfaces of the plate defined by $x_3 = \pm h$ are assumed to be free from stresses. The sketch of the geometry of the plate model is shown through Figure 1. We shall consider the waves along $x_1$-axis propagating with phase velocity $c$. The components of displacement vector $\mathbf{u} = \left(u_1, u_2, u_3\right)$ for the considered Lamb waves, for 2-dimensional problem in $x_1 - x_3$ plane, are as follows

$$\mathbf{u} = \left(u_1, u_2, u_3\right) = \left(u_1\left(x_1, x_3, t\right), \ 0, \ u_3\left(x_1, x_3, t\right)\right) \tag{6}$$

In view of the considered plate model as above and by applying the operator to equations of motion (3), we have

$$\sigma_{11,1} + \sigma_{31,3} = \rho\Gamma \ddot{u}_1, \tag{7a}$$

$$\sigma_{31,1} + \sigma_{33,3} = \rho\Gamma \ddot{u}_3 \tag{7b}$$

Using equations (4) and (5), equations (7) becomes

$$C_{11}\frac{\partial^2 u_1}{\partial^2 x_1} + C_{44}\frac{\partial^2 u_1}{\partial^2 x_3} + (C_{12} + C_{44})\frac{\partial^2 u_3}{\partial x_1 \partial x_3} = \rho\Gamma\frac{\partial^2 u_1}{\partial^2 t} \tag{8a}$$

$$(C_{12} + C_{44})\frac{\partial^2 u_1}{\partial x_1 \partial x_3} + C_{44}\frac{\partial^2 u_3}{\partial^2 x_1} + +C_{11}\frac{\partial^2 u_3}{\partial^2 x_3} = \rho\Gamma\frac{\partial^2 u_3}{\partial^2 t}, \ \Gamma = (1 - (m_0 l_1)^2 \nabla^2) \tag{8b}$$

Time harmonic wave-solution of equations in (8) can be expressed in the form

$$(u_1, u_3) = (U_1, U_3)\, exp(i\xi\alpha x_3)\, exp\big(i\xi(x_1 - ct)\big), \tag{9}$$

where $U_1$ and $U_3$ are constants representing the amplitudes of the wave excitation, α is an unknown parameter, $c$ is the wave speed along $x_1$ and $\xi$ is the wavenumber and $i = \sqrt{-1}$. The circular frequency, $\omega = \xi c$, $\xi$ is the wavenumber. Inserting equation (9) into equations in equation (8), one obtains the following equations in $U_1$ and $U_3$ as

$$\left\{C_{11} - \rho\left[1 + (m_0 l_1)^2 \xi^2 (1 + \alpha^2)\right]c^2 + C_{44}\alpha^2\right\}U_1 + (C_{12} + C_{44})\alpha U_3 = 0 \tag{10a}$$

$$(C_{12} + C_{44})\alpha U_1 + \left\{C_{44} - \rho\left[1 + (m_0 l_1)^2 \xi^2 (1 + \alpha^2)\right]c^2 + C_{11}\alpha^2\right\}U_3 = 0 \tag{10b}$$

For the non-trivial solutions of equations in (10), determinant of the coefficient matrix of these equations must vanish, we obtain a fourth-degree polynomial equation. The four roots, say, $\alpha_n\,(n = 1,2,3,4)$, of this fourth degree polynomial equation are

$$\alpha_{1,2} = \pm\sqrt{\frac{c^2}{C_s{}^2 - (m_0 l_1)^2 \xi^2 c^2} - 1}\,, \alpha_{3,4} = \pm\sqrt{\frac{c^2}{C_L{}^2 - (m_0 l_1)^2 \xi^2 c^2} - 1}\,, \tag{11}$$

where + sign corresponds to the first index, while the − sign corresponds to the second index, here $C_L^2 = \dfrac{C_{11}}{\rho}$ (classical longitudinal velocity) and $C_S^2 = \dfrac{C_{44}}{\rho}$ (shear wave velocity) in the plate. For each value of $\alpha_n$, the amplitude ratio $Q_n$ are obtained from the equations given in (10) as

$$Q_n = \frac{U_{1n}}{U_{3n}} = -\frac{C_S^2 - [1 + (m_0 l_1)^2 \xi^2 (1 + \alpha_n^2)]c^2 + C_L^2 \alpha_n^2}{(C_L^2 - C_S^2)\alpha_n} \tag{12}$$

Making use of equation (12) and superimposing over n, the displacement components given in equation (9) are expressed as

$$\{u_1, u_3\} = \sum_{n=1}^{4}\{Q_n, 1\}U_{3n}\, exp(i\xi\alpha_n x_3)\, exp\big(i\xi(x_1 - ct)\big). \tag{13}$$

Stresses $\sigma_{kl}$ (local) and $\tau_{kl}$ (nonlocal stresses) are related as

$$\tau_{kl} = \left(1 + (m_0 l_1)^2 \nabla^2\right)\sigma_{kl} = \frac{\sigma_{kl}}{1 + (m_0 l_1)^2 \xi^2 \left(1 + \alpha^2\right)} \tag{14}$$

Utilizing (4), (5), (13) and (14), the expressions of nonlocal stresses can be expressed as

$$\{\tau_{13}, \tau_{23}\} = \sum_{n=1}^{4} \{D_{1n}, D_{3n}\} U_{3n} \exp\left(i\xi\alpha_n h\right) \exp\left(i\xi\left(x_1 - ct\right)\right) \tag{15}$$

where

$$D_{1n} = \frac{i\xi C_{44}(\alpha_n Q_n + 1)}{1 + (m_0 l_1)^2 \xi^2 (1 + \alpha_n^2)}, \; D_{3n} = \frac{i\xi(C_{12} Q_n + C_{11}\alpha_n)}{1 + (m_0 l_1)^2 \xi^2 (1 + \alpha_n^2)}, \tag{16}$$

$$Q_2 = -Q_1, \; Q_4 = -Q_3 .$$

## 4. Boundary Conditions and Frequency Equation

To accurately evaluate wave propagation configurations in a finite plate of thickness 2h, the physical boundaries must remain completely traction-free. Assuming the macro-scale external constraints align such that the top and bottom macroscopic faces of the plate ($x_3 = \pm h$) cannot sustain external loads, the actual nonlocal components of the traction vectors must vanish exactly. Mathematically, these macro-structural boundary requirements for a two-dimensional plane-strain formulation in the $(x_1 - x_3)$ plane are specified as:

$$\tau_{13} = \tau_{33} = 0 \text{ at } x_3 = \pm h . \tag{17}$$

Employing equations (15) into (17), one can obtain four homogeneous simultaneous equations in four unknown, namely $U_{31}, U_{32}, U_{33}$ and $U_{34}$ written in matrix form as:

$$\begin{bmatrix} D_{11}E_1 & D_{12}\bar{E}_1 & D_{13}E_3 & D_{14}\bar{E}_3 \\ D_{31}E_1 & D_{32}\bar{E}_1 & D_{33}E_3 & D_{33}\bar{E}_3 \\ D_{11}\bar{E}_1 & D_{12}E_1 & D_{13}\bar{E}_3 & D_{14}E_3 \\ D_{31}\bar{E}_1 & D_{32}E_1 & D_{33}\bar{E}_3 & D_{33}E_3 \end{bmatrix} \begin{bmatrix} U_{31} \\ U_{32} \\ U_{33} \\ U_{34} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{18}$$

$E_n = \exp\left(i\xi\alpha_n h\right), \bar{E}_n = \exp\left(-i\xi\alpha_n h\right).$

Equations in (18) have a non-trivial solution if the determinant of the coefficient matrix vanishes:

$$\begin{vmatrix} D_{11}E_1 & D_{12}\bar{E}_1 & D_{13}E_3 & D_{14}\bar{E}_3 \\ D_{31}E_1 & D_{32}\bar{E}_1 & D_{33}E_3 & D_{33}\bar{E}_3 \\ D_{11}\bar{E}_1 & D_{12}E_1 & D_{13}\bar{E}_3 & D_{14}E_3 \\ D_{31}\bar{E}_1 & D_{32}E_1 & D_{33}\bar{E}_3 & D_{33}E_3 \end{vmatrix} = 0. \tag{19}$$

This is the frequency equation for the plate oscillations. Now from equation (11), using the properties $\alpha_j = -\alpha_{j-1}$, $(j = 2,4)$, from equation(16) we have

$$D_{12} = D_{11},\ D_{14} = D_{13}\ ,\ D_{32} = -D_{31}, D_{34} = -D_{33}\,. \tag{20}$$

### 4.1 Symmetric Vibrations

For symmetric wave modes (see Figure 2), the displacement u is given by

$$u_3(h) = u_3(-h). \tag{21}$$

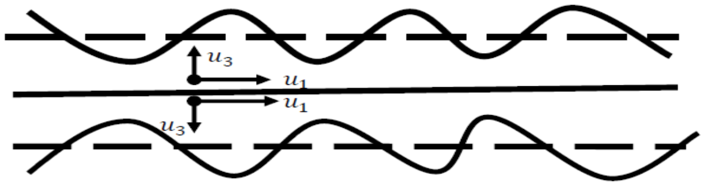

*Figure 2. Symmetric wave*

Substituting equation (21) as $u_3$ in equation (13); it is recognised that,

$$U_{31} = U_{32},\ U_{33} = U_{34}\ . \tag{22}$$

Using the displacement amplitude relationships (22) and (20), equation (19) after algebraic manipulation can be written as

$$\frac{\tan(\xi\alpha_1 h)}{\tan(\xi\alpha_3 h)} = \frac{(\alpha_1 Q_1 + 1)}{(\alpha_3 Q_3 + 1)}\frac{(C_{12}Q_3 + C_{11}\alpha_3)}{(C_{12}Q_1 + C_{11}\alpha_1)}. \tag{23}$$

Simplifying equation (23) using equations (11) and (12), we have

$$\frac{\tan(\xi\alpha_1 h)}{\tan(\xi\alpha_3 h)} = -\frac{(\alpha_1^2 - 1)^2 + \varepsilon^2\xi^2(\alpha_1^4 - 1)(\alpha_3^2 - 1)}{4\alpha_1\alpha_3\left(1 + \varepsilon^2\xi^2(\alpha_1^2 + 1)\right)}, \varepsilon = m_0 l_1\ . \tag{24}$$

Equation (24) characterises the frequency equation of the plate for symmetric modes of vibrations. It can be verified that when $\varepsilon = m_0 l_1 = 0$ that is, when plate is a local (classical) elastic plate, the frequency equation (24) reduces to

$$\frac{\tan(\xi\alpha_1 h)}{\tan(\xi\alpha_3 h)} = -\frac{(\alpha_1^2 - 1)^2}{4\alpha_1\alpha_3}. \tag{25}$$

This equation (25) is same as in Ewing et al. (1957), the frequency equation for symmetric wave modes vibrations of Lamb waves in elastic plate.

### 4.2 Anti-symmetric Vibrations

Similarly, for the anti-symmetric mode of vibrations (see Figure 3), the displacement $u_3$ is given by

$$u_3(h) = -u_3(-h). \tag{26a}$$

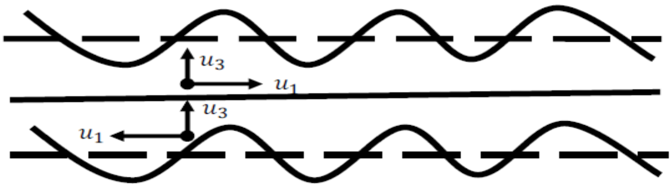

*Figure 3: Anti-Symmetric waves*

Substituting equation (26a) as $u_3$ in equation (13); it is recognised that,

$$U_{31} = -U_{32},\ U_{33} = -U_{34}\ . \tag{26b}$$

In this case, adopting the same procedure as in case of symmetric modes of vibrations, the frequency equation for anti-symmetric plate oscillation is obtained as

$$\frac{\tan(\xi\alpha_1 h)}{\tan(\xi\alpha_3 h)} = -\frac{4\alpha_1\alpha_3\left(1+\varepsilon^2\xi^2\left(\alpha_1{}^2+1\right)\right)}{\left(\alpha_1{}^2-1\right)^2+\varepsilon^2\xi^2\left(\alpha_1{}^4-1\right)\left(\alpha_3{}^2-1\right)}\,. \tag{27}$$

Again, we see that in the absence of nonlocality, that is, when $\varepsilon\left(=m_0 l_1\right)=0$, the equation (27) reduces to

$$\frac{tan(\xi\alpha_1 h)}{tan(\xi\alpha_3 h)} = -\frac{4\alpha_1\alpha_3}{(\alpha_1{}^2-1)^2}. \tag{28}$$

The equation (28) is same as in Ewing et al. (1957), the Lamb wave equation for antisymmetric wave modes of vibrations.

## 5. Analysis at Short and Long Wave Limits

### 5.1 Symmetric Waves

*At short wave limit* for short waves, $\xi h$ *is* very large that is, $\xi h \to \infty$ and wavelengths are short compared with the thickness $2h$ of the plate, consequently the quantities $\xi\alpha_1 h$ and $\xi\alpha_3 h$ are small so long c is finite. Hence taking the short wave limit at $\xi h \to \infty$ in (24) yields

$$\varepsilon^2\xi^2\left(\alpha_1{}^2+1\right)\left(\alpha_1{}^2\alpha_3{}^2+4\alpha_1\alpha_3-\alpha_3{}^2-\alpha_1{}^2+1\right)+\left(\left(\alpha_1{}^2-1\right)^2+4\alpha_1\alpha_3\right)=0\,. \tag{29}$$

This equation (29) is the secular equation for the speed of Rayleigh waves in nonlocal elasticity for an isotropic material. Thus, the limiting speed in this case coincides with the speed of Rayleigh wave in nonlocal elasticity. Here $\alpha_i$ $\left(i=1,3\right)$ are defined in (11) where *c* is the velocity of the Rayleigh wave in the nonlocal elasticity. On simplifying equation (29) for Rayleigh waves in nonlocal elasticity, we obtained

$$\begin{aligned}&\Big[-4C_L^2\left(C_S^2-C_L^2\right)\varepsilon^2\xi^2+16C_S^2\left(C_S^2-C_L^2\right)\varepsilon^6\xi^6-C_L^4+\left(12C_S^4-4C_L{}^4-16C_S^2C_L^2\right)\varepsilon^4\xi^4\Big]c^6\\&+\Big[\left(2C_S^2C_L^2+C_L^4-2C_S^4\right)2\varepsilon^4\xi^4+\left(2C_S^2C_L^2+3C_L^4-2C_S^4\right)\varepsilon^2\xi^2+C_L^4\Big]8C_S^2c^4\\&+\Big[2C_S^2C_L^2-3C_L^4+\left(2C_S^4+2C_S^2C_L^2-4C_L^4\right)\varepsilon^2\xi^2\Big]8C_S^4c^2+16C_S^6C_L^2\left(C_L^2-C_S^2\right)=0\,.\end{aligned}\tag{30}$$

Equation (30) is a sixth degree equation in $c$, and hence it will have six roots, say, $c_j,\left(j=1,2,...6\right)$.These roots are in the form of three pairs of the type$\left(c_j,-\bar{c}_j\right),\left(j=1,2,...6\right)$, where $\bar{c}_j$ is a complex conjugate of$c_j\left(=\pm p_j-iq_j\right)$, $j=1,2,3$. Among these roots, one pair is real, therefore, for this real root, there is no attenuation in the corresponding wave. For other pairs of the roots, there is distinct phase speed $V_j$ and corresponding attenuation coefficient $A_j$ given by

$$V_j=\frac{\left(Re\left(c_j\right)\right)^2+\left(Im\left(c_j\right)\right)^2}{Re\left(c_j\right)},\qquad A_j=\frac{-\omega Re\left(c_j\right)}{\left(Re\left(c_j\right)\right)^2+\left(Im\left(c_j\right)\right)^2},$$

but these two waves travel in opposite direction with same wave speeds.

When $\varepsilon\left(=m_0l_1\right)=0$, expression (29) reduces to $\left(\alpha_1{}^2-1\right)^2+4\alpha_1\alpha_3=0,$ which can be written as

$$\left(\frac{c^2}{C_s{}^2}-1\right)^2+4\sqrt{\frac{c^2}{C_s{}^2}-1}\sqrt{\frac{c^2}{C_L{}^2}-1}=0.\tag{31}$$

This results in agreement with the isotropic case (see Ewing et al. (1957)), where c is the Rayleigh wave phase velocity. This is a well-established Rayleigh wave's equation in the classical elastic medium. In the equation (31), using physical data for aluminium plate, yields $C_R=2975.402\ m/\sec$.

**At long wave limit:** For long waves, $\xi h$ *is* very small that is, $\xi h\rightarrow 0$ and wavelengths are long compared with the thickness $2h$ of the plate, consequently $\xi h\rightarrow 0$ in the equation (**24**) yields

$$\alpha_1\Big[\left(\alpha_1{}^4+4\alpha_3{}^2-2\alpha_1{}^2+1\right)+\varepsilon^2\xi^2\left(\alpha_1{}^2+1\right)\left(\alpha_1{}^2\alpha_3{}^2-\alpha_1{}^2+3\alpha_3{}^2+1\right)\Big]=0,\tag{32}$$

$$c=\frac{C_s{}^2}{\left(1+\varepsilon^2\xi^2\right)}\tag{33}$$

and

$$\left(\alpha_1{}^4+4\alpha_3{}^2-2\alpha_1{}^2+1\right)+\varepsilon^2\xi^2\left(\alpha_1{}^2+1\right)\left(\alpha_1{}^2\alpha_3{}^2-\alpha_1{}^2+3\alpha_3{}^2+1\right)=0.\tag{34}$$

Equation (33) is the limiting velocity of transverse wave, which is dispersive and stimulated with nonlocality parameter. It shows that transverse wave in nonlocal elasticity becomes dispersive and vary with $\varepsilon$. It is also observed that the transverse wave travels slower in nonlocal elastic solid in comparison to the classical (local) solid. On simplification, equation (34) yields

$$c^2 = 4\mathrm{C}_S^2\left(1-\frac{\mathrm{C}_S^2}{\mathrm{C}_L^2}\right)\left[2\varepsilon^2\xi^2\left(1-\frac{\mathrm{C}_S^2}{\mathrm{C}_L^2}\right)+1\right]^{-1}. \tag{35}$$

Equation(35) yields the is the phase velocity of plate waves in nonlocal elastic solid, which becomes dispersive, influenced by the non-locality parameter $\varepsilon$ and travels with less speed than that of in conventional solid material. When $\varepsilon\left(=m_0 l_1\right)=0$, the expression (**35**) reduces to

$$c^2 = 4\mathrm{C}_S^2\left(1-\frac{\mathrm{C}_S^2}{\mathrm{C}_L^2}\right). \tag{36}$$

This result agrees with the classical solid material (see Ewing et al., 1957) apart from notations.

### 5.2 Antisymmetric Waves

The investigation of the wave limits (long and short) in the antisymmetric wave modes is similar to the case of the symmetric wave modes considered in the previous section.

**At short wave limit:** For wave, when wavelengths are short compared with the thickness of the plate, that is, $\xi h \rightarrow \infty$, equation (27) represents the Rayleigh wave equation (29) and the wave propagation converges toward Rayleigh-type waves localized at both of the plate's free surfaces in the nonlocal theory of elasticity.

**At long wave limit (small values of $\xi$):** For wave, when wavelengths are long compared with the thickness of the plate. Equation (27) for the long wave limit yields and neglecting terms of $O\left(\xi h\right)^3$ reduces to the following equation

$$\left(\frac{\xi h}{3}\right)^2\left(\alpha_1^{\,2}-\alpha_3^{\,2}\right)+1=-\frac{4\alpha_3^{\,2}\left(1+\varepsilon^2\xi^2\left(\alpha_1^{\,2}+1\right)\right)}{\left(\alpha_1^{\,2}-1\right)^2+\varepsilon^2\xi^2\left(\alpha_1^{\,4}-1\right)\left(\alpha_3^{\,2}-1\right)}. \tag{37}$$

Equation (37) yields the long flexural waves of the plate in the nonlocal theory of elasticity and is dispersive. It is also observed that the phase velocity decreases and approaches to zero with the wavelength increases and approaches to infinity. In the conventional theory of elasticity, when $\varepsilon\left(=m_0 l_1\right)=0$, above equation (37) reduces to

$$\left(\frac{\xi h}{3}\right)^2 c^2\left(\frac{\mathrm{C}_L^2-\mathrm{C}_S^2}{\mathrm{C}_S^2\,\mathrm{C}_L^2}\right)+1=-4\left(\frac{c^2}{\mathrm{C}_L^2}-1\right)\left(\frac{c^2}{\mathrm{C}_S^2}-2\right)^{-2}, \tag{38}$$

which is in agreement with the classical case of conventional theory of elasticity (see Ewing, et al., 1957).

## 6. Numerical Discussions

In this section, Lamb waves propagation physiognomies in the context of nonlocal elasticity having nonloaclity parameter $\varepsilon\left(=m_0 l_1\right)$ defined in section 2. For numerical computations, aluminium plate is chosen, whose material parameters are given as: Young modulus $E = 69.2\ GPa$, Poisson ratio $\eta = 0.287$, and density $\rho = 2600 kg/m^3$. It is found that inequality $C_R < C_S < C_L$ holds true in the nonlocal elasticity similarly as in conventional (local) theory.

Considering, plate thickness be finite, first, the equation (31) is considered to depict the consequence of nonlocality on the Raleigh waves propagation. Figure 4, it is found that as the nonlocal parameter diminishes, Rayleigh wave velocity approaches $2975.385\ m/sec$. When nonlocal parameter $\varepsilon = 0.01$, at low value region of wavenumber, Rayleigh wave velocity strictly decreases from $2975.385\ m/sec$ (Rayleigh wave velocity in the classical theory of elasticity). When nonlocal parameter, $\varepsilon = 0.001$, Rayleigh wave velocity still decreases strictly with the increase of wavenumber and remain 1.0 to 10 times higher than the previous value. On further decreasing the nonlocal parameter to $\varepsilon = 0.0001$, Rayleigh wave velocity strictly decreases with the increase of wavenumber but remains between 1.0 to 100 strictly decreases with the increase of wavenumber but remains between 1.0 to 100 times higher than the first and 1.0 to 10 times from the second values. It is also witnessed that the Rayleigh wave velocity is not influenced by the $\varepsilon$(nonlocal parameter) if its value $\varepsilon \geq 10^{-4}$. Influence of $\varepsilon$ on the Rayleigh wave velocity is also depicted in Figure 4.

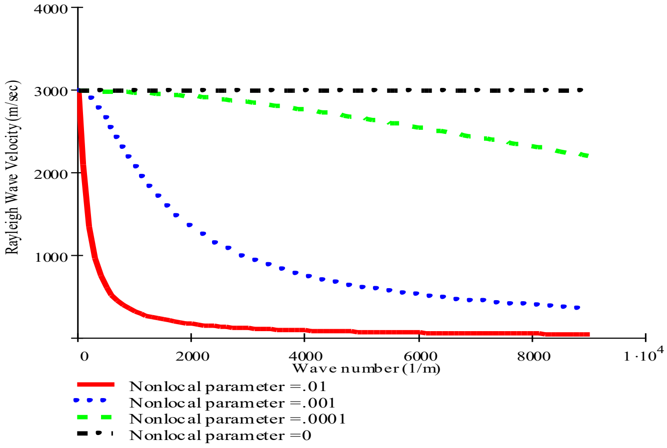

*Figure 4: Variation of Rayleigh wave velocity with wavenumber when nonlocality parameter ε= .01, 0.001, 0.0001 and ε = 0 for an aluminium plate.*

In Figure 5, Rayleigh wave velocity versus nonlocality parameter is plotted and exhibited the variation of Rayleigh wave velocity with the ε. To examine the influence of ε on the propagation of waves, characteristic equations (24) and (27) are considered to study the phase speeds of wave modes. Two types, for in-plane displacement $x_3 (= h)$ such that $u_3\left(h\right) = u_3\left(-h\right)$ (even function) and $u_3\left(h\right) = -u_3\left(-h\right)$ (odd function) solutions emerges, labelled as symmetric and antisymmetric.

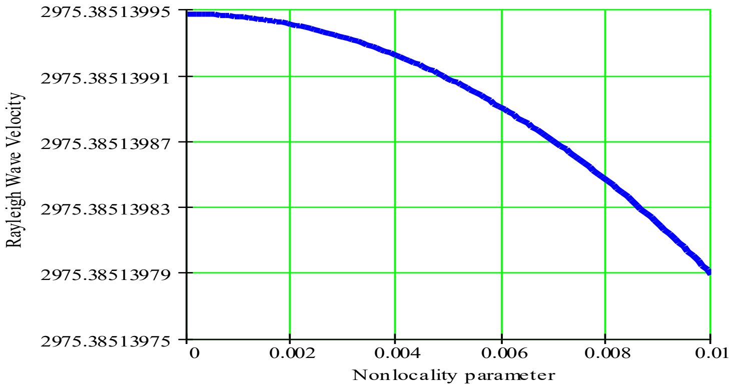

*Figure 5: Variation of Rayleigh wave velocity Vs nonlocality parameter ε= 0 to 0.01 for an aluminium plate.*

Equations (24) and (27) may be viewed as equations involving $c$ (phase velocity), $\varepsilon$ (nonlocality parameter) and $\xi h$ (product of wavenumber and thickness). For given value of$\varepsilon$, both of these equations are transcendental, having infinitely many roots for $c$ in terms of $\xi h$, every root corresponds to a kind of vibration, which are dispersive and the investigation of their behavior is intricate.

Similarly considering characteristic equations (23) and (26) to examine the effect of nonlocality on the waves, the phase speeds of both types of plate wave modes are studied. For,$x_3(=h)$, i.e. $u_3(h)=u_3(-h)$ solutions are named symmetric modes and for $x_3(=h)$, i.e. $u_3(h)=-u_3(-h)$ are named antisymmetric modes emerges. Equations (23) and (26) may be viewed as equations relating $c$ (phase velocity), $\varepsilon$ (nonlocality parameter) and $\xi h$ (product of wavenumber and thickness). For given value of $\varepsilon$, both of these equations, are transcendental, having infinitely many solutions for $c$ in terms of $\xi h$, each solution corresponds to a wave mode of vibration. Thus in both cases of wave modes are clearly dispersive and study of their behaviour in the over-all complex case.

For specified values of wavenumber $\xi$ and $\varepsilon$ (nonlocality parameter), equations (24) or (27) stipulate the allowed values of the velocity c. Significant observations to be made here, firstly, phase velocity $c$ depends on the frequency-thickness product$\xi h$ and $\varepsilon$ (nonlocality parameter), consequently waves dispersive. Secondly, equations (24) and (27) are transcendental and, reliant on the value of $\xi h$ and$\varepsilon$, therefore these equations have infinitely many real solutions. For the allowed velocities, waveforms obtained are known as modes, are naturally categorized into symmetric and antisymmetric types, the extent of allowed modes increases as $\xi h$ increases. In order to evaluate non-local effects on Lamb wave characteristics, both local and non-local dispersion modes are plotted. To study the effect of the parameter $\varepsilon$ on the features of Lamb waves in both theories local and nonlocal), dispersion curves for each mode are exhibited and compared in Figure 6 to Figure 9 for the nonlocality parameter.

Only two fundamental modes—labeled red for symmetric and blue for antisymmetric—exist below a specific threshold frequency. With the increase of $\xi h$ additional modes start appearing, according to the order in which they appear. When$\varepsilon=0$, it represents the local plate mode for the classical case.

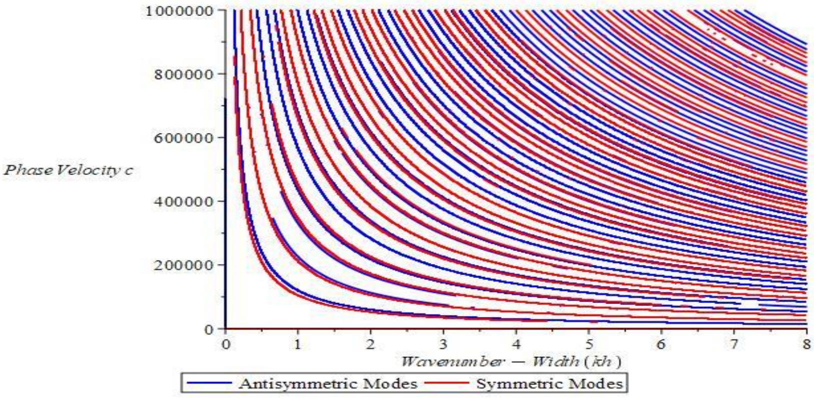

Figure 6. Phase velocity Vs product $\xi h$ (wavenumber-thickness) dispersion curve when nonlocality parameter ε = 0 for an aluminium plate.

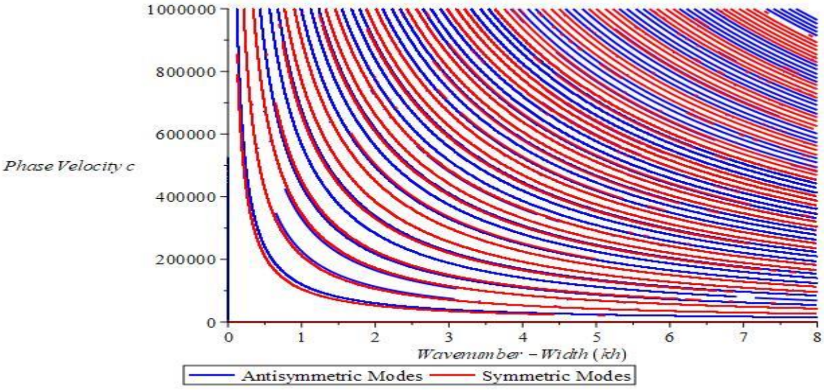

Figure 7. Phase velocity Vs product $\xi h$ (wavenumber-thickness) dispersion curve when nonlocality parameter ε= 0.0001 for an aluminium plate.

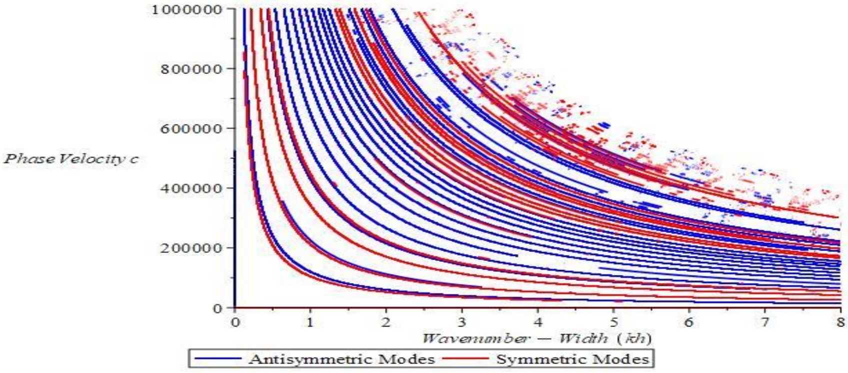

Figure 8. Phase velocity Vs product $\xi h$ (wavenumber-thickness) dispersion curve when nonlocality parameter ε= 0.001 for an aluminium plate.

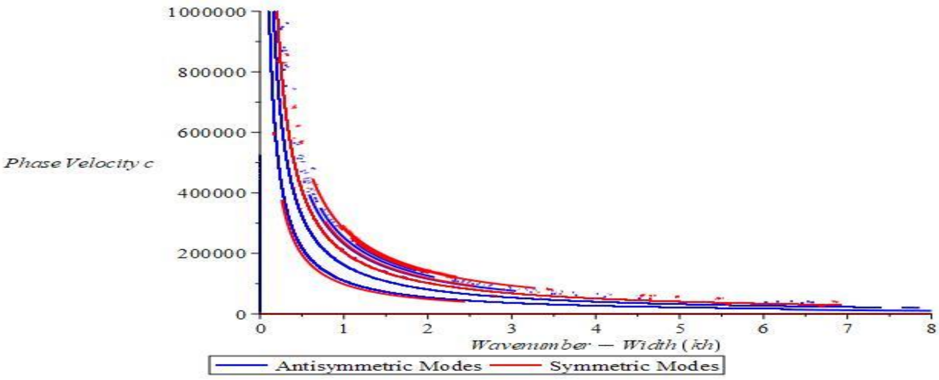

*Figure 9. Phase velocity Vs product $\xi h$ (wavenumber-thickness) dispersion curve when nonlocality parameter ε= 0.01 for an aluminium plate.*

## 7. Conclusion

This paper develops a nonlocal continuum framework to analyze the dispersion of Lamb waves within plates of finite thickness. Propagation of Lamb wave in plates, by modelling them as nonlocal continuum plates is investigated, unlike conventional mechanics, the nonlocal theory, accounts for long-range interatomic forces by describing stress as a function of the strain through the complete volume of the elastic body. Therefore it makes the nonlocal theory predominantly effective for depicting nano-scale structures, where small-scale occurrence insignificant in bulk materials become dominant.

Results show that effects of nonlocality on the mechanical characteristics of very small-sized plates are considerable and as the Lamb waves are broadly designed for non-destructive testing of materials, likely flaws and singularities in the intrinsic characteristics of a plate can be identified at the beginning phase to avoid later stage losses. The widespread usage of Lamb waves for NDT (non-destructive testing) depends on factors such as the capability to propagate without substantial attenuation and dispersion, significant properties of the plate. Therefore the effort is probable to be very suitable in nano-structure designing. The phase velocities of symmetric and antisymmetric waves for the nonlocality effects in the plates are presented. This model can be applied to planar sheets consisting of which are one atom thick, rolled up sheets of graphene just a few nanometres in diameter that can behave as metals. Lamb waves are the ideal tool of ultrasonic excitation to identify and assess defects in nano-plates. At the pre-assigned wave frequency, plate thickness and elastic constants, waves in aluminium plate are depicted.

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